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1994 J. Phys. A: Math. Gen. 27 1781

(http://iopscience.iop.org/0305-4470/27/5/037)

J. Phys. A: Math. Gen. 27 (1994) 1781–1786. Printed in the UK

One-loop renormalization of Chern–Simons field theory in operator regularization

W F Chen†‡§ and Z Y Zhu†‡

† CCAST(World Laboratory), PO Box 8730, Beijing, 100080, Peoples's Republic of China
‡ Institute of Theoretical Physics, Academia Sinica, PO Box 2735, Beijing, 100080, Peoples's Republic of China
§ International Center for Theoretical Physics, PO Box 586, 34100, Trieste, Italy

Received 15 November 1993

Abstract. Operator regularization, together with the background field method, is used to calculate the one-loop renormalization constant of Chern–Simons field theory. The result shows the existence of the famous k shift, i.e. $k \longrightarrow k + \mathrm{sgn}(k)C_V$.

In this paper, we adopt operator regularization with the background field method to calculate one-loop renormalization of perturbative Chern–Simons field theory, which preserves the explicit quantum gauge symmetry [1, 2]. As we all know, this problem has been paid much attention for years [3–13]. The results seem to depend on regularization schemes, some of them show the existence of the k shift, i.e. $k \longrightarrow k + \mathrm{sgn}(k)C_V$ [3–9, 13], and others do not [10–13]. One conjecture is that the result depends on the gauge invariance of the regularization. Our result shows the existence of the k shift in operator regularization.

The non-Abelian classical Chern–Simons action is

$$S[\mathcal{A}] = \frac{k}{4\pi}\int_M \mathrm{Tr}(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \tag{1}$$

where $\mathcal{A} = \mathcal{A}^a_\mu T^a\,\mathrm{d}x^\mu$, T^a are representation matrices of $SU(N)$ generators. The normalization we take is $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The parameter k must be an integer in order to make the quantum theory gauge invariant. Making the replacement

$$g^2 = 4\pi/|k| \qquad \mathcal{A} \longrightarrow g\mathcal{A} \tag{2}$$

we can express the classical action as

$$S_{\mathrm{r}}[\mathcal{A}] = \mathrm{sgn}(k)\int \mathrm{Tr}(\mathcal{A}\wedge \mathrm{d}\mathcal{A} + \tfrac{2}{3}g\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}). \tag{3}$$

As is done in the background field method [2], we write $\mathcal{A}$ as $\mathcal{A}(x) = A(x) + Q(x)$, where $A(x)$ is the background field, i.e. satisfies $\delta S[A]/\delta A = 0$, and $Q(x)$ is the quantum field. Choosing the covariant background gauge condition $D^{\mu ab}[A]Q^b_\mu = 0$, $D^{ab}_\mu[A] = \partial_\mu\delta^{ab} + gf^{acb}A^c_\mu$, and using the standard Faddeev–Popov technique, one finds the generating functional of Green functions

$$Z[A] = \int \mathcal{D}Q\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\exp\mathrm{i}\left\{S_{\mathrm{r}}[A+Q] - \int(\mathrm{d}x)\,\frac{1}{2\alpha}(D_\mu[A]Q^{\mu a}(x))^2 - \int(\mathrm{d}x)\,\bar{c}^a(x)\,D^{ac}_\mu[A]\,D^{\mu cb}[A+Q]\,c^b(x)\right\}. \tag{4}$$

The one-loop quantum generating functional is

$$Z_1[A] = \int \mathcal{D}Q\,\mathcal{D}c\,\mathcal{D}\bar{c}\,\exp \mathrm{i}S_\mathrm{r}[A] \exp \mathrm{i} \int (\mathrm{d}x) \bigg\{ Q^a_\mu \bigg(\mathrm{sgn}(k)[-\tfrac{1}{2}\epsilon^{\mu\nu\rho}\partial_\rho\delta^{ab} + \tfrac{1}{2}g\epsilon^{\mu\nu\rho}f^{abc}A^c_\rho] + \frac{1}{2\alpha}D^{\mu a}[A]D^{\nu b}[A]\bigg) Q^b_\nu + \bar{c}^a D^{ac}_\mu[A]D^{\mu cb}[A]c^b \bigg\}$$

$$= \exp \mathrm{i}S_\mathrm{r}[A] \frac{\det(D^{\mu ac}[A]D^{cb}_\mu[A])}{\det^{1/2}\{\mathrm{sgn}(k)[-\epsilon^{\mu\nu\rho}\partial_\rho\delta^{ab} + g\epsilon^{\mu\nu\rho}f^{abc}A^c_\rho] + (1/\alpha)D^{\mu ac}[A]D^{\nu cb}[A]\}}. \tag{5}$$

As in [1], we adopt operator regularization to evaluate the determinants. For an elliptic operator $H = H_0 + H_I$, we have that

$$\det(H) = \exp \mathrm{Tr}\ln(H) = \exp \mathrm{Tr} \lim_{s\to 0} \left\{ \frac{\mathrm{d}^m}{\mathrm{d}s^m} \left[\frac{s^{m-1}}{m!} \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \exp(-Ht) \right] \right\} \tag{6}$$

and

$$\mathrm{Tr}\exp(-Ht) = \mathrm{Tr}\,\mathrm{e}^{-(H_0+H_I)t}$$
$$= \mathrm{Tr}\bigg[\mathrm{e}^{-H_0t} + (-t)\mathrm{e}^{-H_0t}H_I + \frac{(-t)^2}{2}\int_0^1 \mathrm{d}u\, \mathrm{e}^{-(1-u)H_0t}H_I\mathrm{e}^{-uH_0t}H_I + \frac{(-t)^3}{3}\int_0^1 \mathrm{d}u\, u \int_0^1 \mathrm{d}v\, \mathrm{e}^{-(1-u)H_0t}H_I\mathrm{e}^{-u(1-v)H_0t}H_I\mathrm{e}^{-uvH_0t}H_I + \cdots \bigg] \tag{7}$$

where H_0 is defined to be the part of H independent of the background field A and H_I is a polynomial of the A field. From (6) and (7), one can see that H_0 must be an elliptic operator in order to ensure convergence of the t integrals. Furthermore, we observe that, for Chern–Simons field theory, choosing $m = 1$ is sufficient to make the integral UV convergent.

From (5), it is easy to see that the H_0 part of the ghost determinant operator is obviously positive-definite after a Wick rotation. The operator in the denominator is also well behaved. For the sake of calculational convenience, we take $\det H = [\det H^2]^{1/2}$. Thus (5) may be expressed as

$$Z_1[A] = \exp \mathrm{i}S_\mathrm{r}[A] \exp\left\{ \lim_{s\to 0}\left[-\frac{\mathrm{d}}{\mathrm{d}s}G(s)\right] - \lim_{s\to 0}\left[-\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}s}B(s)\right]\right\} \tag{8}$$

where

$$G(s) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}\,\mathrm{Tr}\exp(D^{ac}_\mu[A]D^{\mu cb}[A]t)$$
$$= G_{AA}(s) + G_{AAA}(s) + \cdots$$
$$B(s) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}\,\mathrm{Tr}\exp\big[\mathrm{sgn}(k)(-\epsilon^{\mu\nu\rho}\partial_\rho\delta^{ab} + g\epsilon^{\mu\nu\rho}f^{abc}A^c_\rho) + (1/\alpha)D^{\mu ac}[A]D^{\nu cb}[A]\big]^2 \tag{9}$$
$$= B_{AA}(s) + B_{AAA}(s) + \cdots.$$

When we calculate the two-point function, we only need to consider $G_{AA}(s)$ and $B_{AA}(s)$, i.e. the terms quadratic in A fields in the expansion of $G(s)$ and $B(s)$. Let us first see the contribution from the ghost part

$$G_{AA}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1}\, \mathrm{Tr} \Bigg\{ (-t)\mathrm{e}^{-p^2 t\delta^{ab}} \left[-g^2 f^{cde} f^{ekl} A_\mu^d(x) A^{\mu k}(x) \right] + \frac{(-t)^2}{2} \int_0^1 \mathrm{d}u\, \mathrm{e}^{-(1-u)p^2 t\delta^{ab}} \times \left[\mathrm{i}g f^{cdp}(p^\mu A_\mu^p + A_\mu^p p^\mu) \mathrm{e}^{-up^2 t\delta^{mn}} \mathrm{i}g f^{ghr}(p^\nu A_\nu^r + A_\nu^r p^\nu) \right] \Bigg\}. \tag{10}$$

Introducing a complete orthonormal set $|p\rangle$, the eigenstates of the operator p_μ, and making use of the following relations in three dimensions

$$\langle x|p\rangle = \frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^{3/2}} \qquad \int \mathrm{d}^3p\, |p\rangle\langle p| = 1$$
$$\langle p|A(x)|q\rangle = A(p-q)\mathrm{e}^{-\mathrm{i}x(p-q)} \qquad A(p-q) = \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} A(x) \mathrm{e}^{-\mathrm{i}x(p-q)} \tag{11}$$

we find that

$$G_{AA}(s) = \frac{1}{\Gamma(s)} C_V g^2 \delta^{ab} \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}t\, t^{s-1} \int \mathrm{d}^3p\, \mathrm{d}^3q \times \Bigg[(-t)\mathrm{e}^{-p^2 t} A_\mu^a(p-q) A^{\nu b}(q-p) + \frac{t^2}{2} \int_0^1 \mathrm{d}u\, \mathrm{e}^{-[(1-u)p^2+uq^2]t} (p_\mu + q_\mu)(p_\nu + q_\nu) A^{\mu a}(p-q) A^{\nu b}(q-p) \Bigg] \tag{12}$$

where C_V is the quadratic Casimir operator in the adjoint representation of the gauge group. Shifting the integration variable $p \to p+q$, we can show that the first term in (12) vanishes, so that

$$G_{AA}(s) = \frac{1}{\Gamma(s)} \frac{C_V g^2}{2} \delta^{ab} \int_0^\infty \mathrm{d}t\, t^{s+1} \int_0^1 \mathrm{d}u \int \frac{\mathrm{d}^3p\, \mathrm{d}^3q}{(2\pi)^3} \exp\{-[(1-u)p^2 + uq^2]t\} \times (p_\mu + q_\mu)(p_\nu + q_\nu) A^{\mu a}(p) A^{\nu b}(-p). \tag{13}$$

Changing the integration variable $q \to q + (1-u)p$ and using

$$\int \frac{\mathrm{d}^nq}{(2\pi)^n} \frac{(q^2)^r}{(q^2+c^2)^n} = \frac{1}{(16\pi^2)^{n/4}} (c^2)^{n/2+r-m} \frac{\Gamma(r+n/2)\Gamma(m-r-n/2)}{\Gamma(n/2)\Gamma(m)}$$
$$\int_0^1 \mathrm{d}u\, u^{m-1}(1-u)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{14}$$

we find that

$$G_{AA}(s) = \frac{1}{\Gamma(s)}\frac{C_V g^2}{2}\delta^{ab}\int d^3p\left\{A^{\mu a}(p)A^{\nu b}(-p)\right.$$
$$\left.\times\int_0^1 du\int\frac{d^3q}{(2\pi)^3}[q^2\delta_{\mu\nu}+(1-2u)^2p_\mu p_\nu]\int_0^\infty dt\, t^{s+1}e^{-[q^2+u(1-u)p^2]t}\right\}$$
$$= \frac{\Gamma(s+2)}{\Gamma(s)}\frac{C_V g^2}{2}\int d^3p\left\{A^{\mu a}(p)A^{\nu b}(-p)\right.$$
$$\left.\times\int_0^1 du\int\frac{d^3q}{(2\pi)^3}[q^2\delta_{\mu\nu}+(1-2u)^2p_\mu p_\nu]\frac{1}{[q^2+u(1-u)p^2)]^{s+2}}\right\}$$
$$= \frac{\Gamma(s+2)}{\Gamma(s)}\frac{C_V g^2}{2}\frac{1}{8\pi^{3/2}}\int d^3p\left\{A^{\mu a}(p)A^{\nu b}(-p)\right.$$
$$\times\int_0^1 du\left[\delta_{\mu\nu}[u(1-u)p^2]^{1/2-s}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+2)\Gamma(s)}\right.$$
$$\left.\left.+p_\mu p_\nu(1-2u)^2[u(1-u)p^2]^{-1/2-s}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\right]\right\}$$
$$= \frac{C_V g^2}{8}\frac{1}{8\pi^{3/2}}\int d^3p\, A^{\mu a}(p)A^{\nu b}(-p)\frac{1}{(p^2)^{1/2+s}}(p^2\delta_{\mu\nu}-p_\mu p_\nu)$$
$$\times\frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{3}{2}-s)\Gamma(\frac{1}{2}+s)}{\Gamma(s+2)\Gamma(3-2s)}. \tag{15}$$

For $B_{AA}(s)$, after a lengthy calculation, we finally find that

$$B_{AA}(s) = -\frac{C_V g^2}{2}\frac{1}{8\pi^3/2}\int d^3p\left[A^{\mu a}(p)A^{\nu b}(-p)\frac{1}{(p^2)^{1/2+s}}(p^2\delta_{\mu\nu}-p_\mu p_\nu)\right.$$
$$\left.\times\frac{\Gamma(\frac{1}{2}-s)\Gamma(\frac{3}{2}-s)\Gamma(\frac{1}{2}+s)}{\Gamma(s+2)\Gamma(3-2s)}\right]$$
$$+C_V g^2\frac{1}{4\pi^{3/2}}\alpha^s\,\mathrm{sgn}(k)\frac{\Gamma(s+2)\Gamma(\frac{1}{2}-s)\Gamma(1-s/2)\Gamma(s/2)}{\Gamma(s+1)\Gamma^2(s)\Gamma(1+s/2)\Gamma(\frac{3}{2}-s)}$$
$$\times\int d^3p\, A_\mu^a(p)\delta^{ab}(p^2)^{-s/2}\epsilon^{\mu\nu\rho}p_\rho A_\nu^b(-p) \tag{16}$$

where we have used the properties of the projection operator and Feynman integral parametrization

$$e^{-(p^2\delta_{\mu\nu}-p_\mu p_\nu)t} = e^{-p^2t}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right) \qquad e^{-(1/\alpha^2)p^2tp_\mu p_\nu} = e^{-1/\alpha^2p^4t}\frac{p_\mu p_\nu}{p^2}$$

$$\frac{1}{A^{km}B^{ln}} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\int_0^1\frac{x^{m-1}(1-x)^{n-1}}{[A^kx+B^l(1-x)]^{m+n}} \tag{17}$$
$$= \frac{\Gamma(km+ln)}{\Gamma(km)\Gamma(ln)}\int_0^1\frac{x^{km-1}(1-x)^{ln-1}}{[Ax+B(1-x)]^{m+n}}.$$

From (8), (15) and (16), we know that the one-loop quantum correction to the two-point function of the background fields is

$$\lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathrm{sgn}(k)C_V g^2\frac{1}{4\pi^{3/2}}\alpha^s\frac{\Gamma(s+2)\Gamma(\frac{1}{2}-s)\Gamma(1-s/2)\Gamma(s/2)}{\Gamma(s+1)\Gamma^2(s)\Gamma(1+s/2)\Gamma(\frac{3}{2}-s)}\right.$$
$$\left.\times\int \mathrm{d}^3p\, A^a_\mu(p)\delta^{ab}(p^2)^{-s/2}\epsilon^{\mu\nu\rho}p_\rho A^b_\nu(-p)\right]$$
$$=\mathrm{sgn}(k)\frac{C_V}{8\pi}g^2\int \mathrm{d}^3p\, A^a_\mu(p)\delta^{ab}\epsilon^{\mu\nu\rho}p_\rho A^b_\nu(-p)$$
$$=\mathrm{i}\,\mathrm{sgn}(k)\frac{C_V}{4\pi}g^2\int \mathrm{d}^3x\,\tfrac{1}{2}\epsilon^{\mu\nu\rho}A^a_\mu(x)\partial_\nu A^a_\rho(x)$$
$$=\mathrm{i}\frac{C_V g^2}{4\pi}\mathrm{sgn}(k)\int \mathrm{Tr}(A\wedge \mathrm{d}A) \tag{18}$$

where we have taken $\Gamma(as)|_{s\to 0}\approx 1/\alpha s$ and $\Gamma'(s)|_{s\to 0}\approx -1/s^2$. Therefore the one-loop wavefunction renormalization constant is

$$Z_A = 1+\frac{C_V}{4\pi g^2}. \tag{19}$$

The quantum correction to the three-point function of the background fields can be obtained directly with the aid of the explicit gauge symmetry in the background field method. This means that

$$Z_g = Z_A{}^{-1/2} \tag{20}$$

so the one-loop quantum corrected three-point function is

$$\mathrm{i}g\left(1+\frac{k}{4\pi}g^2\right)\mathrm{sgn}(k)\int \mathrm{d}^3p\,\mathrm{d}^3q\,\tfrac{1}{2}A^{\mu a}(p)A^{\nu b}(q)A^{\rho c}[-(p+q)]f^{abc}\epsilon_{\mu\nu\rho}$$
$$=\mathrm{i}\left(1+\frac{C_V}{4\pi}g^2\right)\mathrm{sgn}(k)\int \mathrm{d}^3x\,\tfrac{1}{2}gf^{abc}\epsilon_{\mu\nu\rho}A^{\mu a}(p)A^{\nu b}(q)A^{\rho c}[-(p+q)]$$
$$=\mathrm{i}\left(1+\frac{C_V}{4\pi}g^2\right)\mathrm{sgn}(k)\int \mathrm{Tr}\,\tfrac{2}{3}gA\wedge A\wedge A. \tag{21}$$

From (5), (18) and (21), we have that, up to one loop

$$Z_1[A]=\exp\left[\mathrm{i}\left(1+\frac{C_V}{4\pi}g^2\right)\mathrm{sgn}(k)\int \mathrm{Tr}(A\wedge \mathrm{d}A+\tfrac{2}{3}gA\wedge A\wedge A)\right]. \tag{22}$$

Considering the classical action (3) and the scale replacement (2), we find that

$$Z_1[A]=\exp\mathrm{i}\left[\left(\frac{1}{g^2}+\frac{C_V}{4\pi}\right)\mathrm{sgn}(k)\int \mathrm{Tr}(A\wedge \mathrm{d}A+\tfrac{2}{3}gA\wedge A\wedge A)\right]$$
$$=\exp\left\{\mathrm{i}\frac{k+\mathrm{sgn}(k)C_V}{4\pi}S_{\mathrm{cs}}[A]\right\} \tag{23}$$

and hence the one-loop effective action

$$\Gamma_1[A]=\frac{k+\mathrm{sgn}(k)C_V}{4\pi}\int \mathrm{Tr}(A\wedge \mathrm{d}A+\tfrac{2}{3}A\wedge A\wedge A). \tag{24}$$

In summary, we have used the background field method and operator regularization to calculate the one-loop renormalization of Chern–Simons field theory, and find the theory is finite at one loop, and that the quantum correction only appears in the shift of the coupling constant $k\longrightarrow k+\mathrm{sgn}(k)C_V$.

Acknowledgments

One of the authors (WFC) would like to thank Professor Abdus Salam and all the staff at ICTP for hospitality. This work is partially supported by grant LWTZ-1298 of the Chinese Academy of Science and the National Natural Science Foundation of China.

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